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Bilinearization of $N = 1$ supersymmetric Korteweg–de Vries equation revisited

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Abstract

We consider the $N = 1$ supersymmetric Korteweg–de Vries (sKdV) equation within the framework of Hirota's bilinear method. We construct a Bäcklund transformation which may be interpreted as the modified sKdV equation. Also, we find a Lax representation and a nonlinear superposition formula. By direct applications of the nonlinear superposition formula, we calculate soliton solutions for the sKdV equation.

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1. Introduction

The $N = 1$ supersymmetric Korteweg–de Vries (sKdV) equation is introduced by Manin and Radul [10] in their theory on supersymmetric KP hierarchy. It reads

$$\Phi_t - 3(\Phi\mathcal{D}\Phi)_x + \Phi_{xxx} = 0, \tag{1}$$

where Φ is a Grassmann odd variable depending on super spatial variables (x, θ) and temporal variable t. $\mathcal{D} = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$ is the super derivative. Since the work of Manin and Radul, this sKdV equation has attracted much attention in the community of mathematical physics. It is shown that this equation, like its classical counterpart KdV equation, has many interesting properties, such as bi-Hamiltonian structures [16], infinite number of symmetries, Painlevé property [12], etc.

Apart from algebraic and geometrical properties, integrable equations possess interesting solutions known as multi-solitons. In the theory of integrable systems, there are many effective methods for constructing solutions, such as inverse scattering transform, Darboux transformation, Bäcklund transformation (BT), Hirota's bilinear approach and symmetry methods. The study of Darboux transformation for supersymmetric integrable systems was initiated in [5] and continued in [6]. As the consequence of applying Darboux transformation, one kind of soliton is obtained for sKdV equation, which is characterized by the appearance of certain constraints on the parameters. Recently, Hirota's method (see [3, 4] for example) was used in the context of the sKdV equation in [13] and the explicit solutions were calculated within this framework by Carstea, Ramani and Grammaticos [2]. The appealing feature here is that these solutions are free of any constraint. Very recently, in [9] a BT and the associated nonlinear superposition formula are found. Since this nonlinear superposition formula is differential and algebraic in nature, it provides us a very useful tool to calculate more complicated solutions systematically.

The purpose of the present paper is to examine the sKdV equation from the viewpoint of Hirota's method. We will show that a bilinear BT can be worked out. This bilinear BT in turn can be regarded as a integrable equation, namely a modified sKdV equation. Furthermore, we found a new Lax representation for the sKdV equation. We also present a nonlinear superposition formula, which allows us to calculate soliton solutions straightforwardly. All these results are the consequence of bilinerization of the sKdV equation.

The paper is organized as follows. In the next section, we construct a BT for the sKdV. In section 3, we show that a Lax representation and a modified sKdV equation can be derived from this BT. Section 4 is devoted to the bilinear superposition formula. We conclude the paper with section 5.

2. Bäcklund transformation

Following [1, 13], we now transform the sKdV equation (1) into Hirota's bilinear form. To this end, we introduce

$$\Phi = \Psi_x,$$

then the sKdV equation (1) is reformed into its potential form

$$\Psi_t - 3\Psi_x \mathcal{D}\Psi_x + \Psi_{xxx} = 0, \tag{2}$$

then through the following substitution [1, 13]

$$\Psi = -2\mathcal{D}(\ln f), \tag{3}$$

equation (2) is brought to

$$S\left(D_t + D_x^3\right) f \cdot f = 0, \tag{4}$$

where Hirota's derivatives are defined as follows:

$$SD_t^m D_x^n f \cdot g = \left(\mathcal{D}_{\theta_1} - \mathcal{D}_{\theta_2}\right)\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\right)^m \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^n f(x_1, t_1, \theta_1) g(x_2, t_2, \theta_2)\Big|_{\substack{x_1=x_2\\ t_1=t_2\\ \theta_1=\theta_2}}.$$

We now have the following proposition.

Proposition 1. *Let f be a solution of equation (4) and g is associated with f via the following equations,*

$$(SD_x - \lambda S) f \cdot g = 0, \tag{5}$$

$$\left(D_t + 3\lambda^2 D_x - 3\lambda D_x^2 + D_x^3\right) f \cdot g = 0, \tag{6}$$

where λ is an arbitrary (bosonic) constant. Then, g is another solution of (4).

Proof. We consider

$$\mathbb{P} = \left[S\left(D_t + D_x^3\right) f \cdot f\right] gg - ff\left[S\left(D_t + D_x^3\right) g \cdot g\right]$$

we will show that $\mathbb{P} = 0$.

Indeed, we have

$$\begin{aligned}\mathbb{P} &\overset{(A.1-A.2)}{=} 2S(D_t f\cdot g)\cdot gf + 2S\left(D_x^3 f\cdot g\right)\cdot fg + 6S\left(D_x^2 f\cdot g\right)\cdot (D_x g\cdot f)\\ &\quad - 3\left[(SD_x f\cdot f)\left(D_x^2 g\cdot g\right) - (SD_x g\cdot g)\left(D_x^2 f\cdot f\right)\right]\\ &\overset{(A.3)}{=} 2S\left(\left(D_t + D_x^3\right) f\cdot g\right)\cdot gf + 6D_x(SD_x f\cdot g)\cdot (D_x g\cdot f)\\ &\overset{(5)}{=} 2S\left(\left(D_t + D_x^3\right) f\cdot g\right)\cdot gf + 6\lambda D_x(Sf\cdot g)\cdot (D_x g\cdot f)\\ &\overset{(A.4)}{=} 2S\left[\left(D_t - 3\lambda D_x^2 + D_x^3\right) f\cdot g\right]\cdot fg + 6\lambda D_x(SD_x f\cdot g)\cdot (gf)\\ &\overset{(A.5)}{=} 2S\left[\left(D_t - 3\lambda D_x^2 + 3\lambda^2 D_x + D_x^3\right) f\cdot g\right]\cdot fg\\ &\quad + 6D_x((SD_x - \lambda S)f\cdot g)\cdot (D_x g\cdot f) \overset{(5-6)}{=} 0.\end{aligned}$$

Thus, our proposition is proved. □

Next we convert the bilinear BT to ordinary form. To this end, let

$$\Psi = -2\mathcal{D}(\ln f), \qquad \bar{\Psi} = -2\mathcal{D}(\ln g)$$

then we find that

$$\begin{aligned}&(\Psi + \bar{\Psi})_x = \lambda(\Psi - \bar{\Psi}) + \tfrac{1}{2}(\Psi - \bar{\Psi})(\mathcal{D}\Psi - \mathcal{D}\bar{\Psi}),\\ &\mathcal{D}^{-1}(\Psi_t - \bar{\Psi}_t) = 3\lambda\left[\mathcal{D}(\Psi + \bar{\Psi})_x - \tfrac{1}{2}(\mathcal{D}\Psi - \mathcal{D}\bar{\Psi})^2\right] - 3\lambda^2(\mathcal{D}\Psi - \mathcal{D}\bar{\Psi})\\ &\qquad - \mathcal{D}(\Psi - \bar{\Psi})_{xx} - \tfrac{1}{4}(\mathcal{D}\Psi - \mathcal{D}\bar{\Psi})^3 + \tfrac{3}{2}(\mathcal{D}\Psi - \mathcal{D}\bar{\Psi})(\mathcal{D}\Psi_x + \mathcal{D}\bar{\Psi}_x).\end{aligned}$$

or

$$\begin{aligned}&(\Psi + \bar{\Psi})_x = \lambda(\Psi - \bar{\Psi}) + \tfrac{1}{2}(\Psi - \bar{\Psi})(\mathcal{D}\Psi - \mathcal{D}\bar{\Psi}),\\ &(\Psi - \bar{\Psi})_t = 3\lambda[(\Psi + \bar{\Psi})_{xx} - (\mathcal{D}\Psi - \mathcal{D}\bar{\Psi})(\Psi_x - \bar{\Psi}_x)] - 3\lambda^2(\Psi_x - \bar{\Psi}_x)\\ &\qquad - \tfrac{3}{4}(\mathcal{D}\Psi - \mathcal{D}\bar{\Psi})^2(\Psi_x - \bar{\Psi}_x) + \tfrac{3}{2}(\Psi - \bar{\Psi})_x(\mathcal{D}\Psi + \mathcal{D}\bar{\Psi})_x\\ &\qquad + \tfrac{3}{2}(\mathcal{D}\Psi - \bar{\Psi})_{xx} - (\Psi - \bar{\Psi})_{xxx},\end{aligned}$$

this constitutes our BT in ordinary variables, which is that found in [9] by adopting a different procedure.

3. Lax representation and modified equation

In the theory of classical integrable systems, a BT is not only an effective procedure to construct particular solutions, it also contains other importance information on a given system. For example, it is often possible to derive a Lax representation. Furthermore, a BT itself can be interpreted as an integrable system. We now show that it is also the case for the sKdV equation.

We first show that our BT (5) and (6) obtained in the previous section also supplies us a Lax representation for the sKdV equation (1). To this end, suppose

$$\Sigma = \frac{g}{f},$$

then the variables f and g can be eliminated from equations (5) and (6). The elimination results in

$$\mathcal{D}\Sigma_x - \Psi_x\Sigma + \lambda\mathcal{D}\Sigma = 0, \tag{7}$$

$$\Sigma_t - 3\lambda(\mathcal{D}\Psi_x)\Sigma + 3(\lambda^2 - (\mathcal{D}\Psi_x))\Sigma_x + 3\lambda\Sigma_{xx} + \Sigma_{xxx} = 0. \tag{8}$$

Therefore we have

Proposition 2. *The compatibility condition of (7) and (8) is the sKdV equation (1).*

Proof. Direct calculations.

It is interesting to observe that the linear problem (7) allows us to have the following Lax operator:

$$L = \partial - \Phi \mathcal{D}^{-1}.$$

We also note that this Lax operator is different from that considered in [11]. Indeed, the present Lax operator is in the form of a modified constrained system on the one hand. On the other hand, we may consider the linear problem (7) as a reduction of the more general energy dependent super Hill problem studied in [7]. With this Lax operator we may construct the sKdV hierarchy through the standard fractional power method

$$\frac{\partial L}{\partial t_n} = [L, (L^n)_{\geqslant 1}].$$

Now let us view the BT (5) and (6) as a system, which should be an integrable system. By means of

$$\Upsilon = \ln(g/f),$$

we convert (5) into

$$\Phi = \mathcal{D}\Upsilon_x + (\mathcal{D}\Upsilon)\Upsilon_x \tag{9}$$

and (6) into

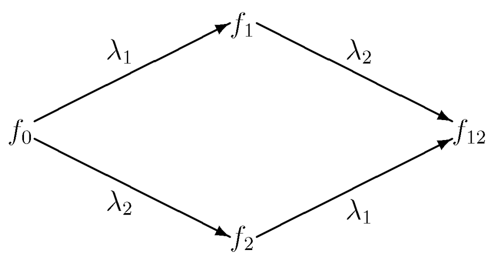
$$\Upsilon_t + 3\lambda^2\Upsilon_x + \Upsilon_{xxx} - 2\Upsilon_x^3 + 3(\lambda + \Upsilon_x)(\mathcal{D}\Upsilon)(\mathcal{D}\Upsilon_x) = 0. \tag{10}$$

We see that equation (9) is the Miura-type transformation found in [11, 17], while equation (10) is a generalization of the supersymmetric modified KdV equation. If we set the parameter λ to zero, we recover the standard version, which was discussed recently in [8] according to Hirota's method. □

4. Nonlinear superposition formula

In this section, we will show that a nonlinear superposition formula associated with the bilinear BT (5) and (6) can be derived naturally.

Let us start with an arbitrary solution f_0. By taking BT with parameter λ_1, we have another solution f_1. Then we do the second step of BT starting with f_1 and parameter taking value λ_2. The solution obtained in this step is denoted as f_{12}. Now we exchange the parameters, namely starting with f_0 and doing the first step of BT with parameter λ_2, we obtain a solution f_2. Then doing the second step of BT with the parameter λ_1, we arrive at a solution f_{21}. By Bianchi's permutation theorem, we may take $f_{12} = f_{21}$. This process is represented schematically by



Bianchi's diagram

Let us list the equations resulting from the above procedure. From (6), we have

$$\left(D_t + 3\lambda_1^2 D_x - 3\lambda_1 D_x^2 + D_x^3\right) f_0 \cdot f_1 = 0, \tag{11}$$

$$\left(D_t + 3\lambda_2^2 D_x - 3\lambda_2 D_x^2 + D_x^3\right) f_1 \cdot f_{12} = 0, \tag{12}$$

$$\left(D_t + 3\lambda_2^2 D_x - 3\lambda_2 D_x^2 + D_x^3\right) f_0 \cdot f_2 = 0, \tag{13}$$

$$\left(D_t + 3\lambda_1^2 D_x - 3\lambda_1 D_x^2 + D_x^3\right) f_2 \cdot f_{12} = 0, \tag{14}$$

from equations (11) and (14), we have

$$\begin{aligned}(D_t f_0 \cdot f_1) f_2 f_{12} - (D_t f_2 \cdot f_{12}) f_0 f_1 = {} & -3\lambda_1^2 [(D_x f_0 \cdot f_1) f_2 f_{12} - f_0 f_1 (D_x f_2 \cdot f_{12})] \\ & + 3\lambda_1 \left[\left(D_x^2 f_0 \cdot f_1\right) f_2 f_{12} - f_0 f_1 \left(D_x^2 f_2 \cdot f_{12}\right]\right. \\ & - \left(D_x^3 f_0 \cdot f_1\right) f_2 f_{12} + f_0 f_1 \left(D_x^3 f_2 \cdot f_{12}\right),\end{aligned} \tag{15}$$

by means of the identity (A.6), we obtain from above

$$\begin{aligned}(D_t f_0 \cdot f_2) f_1 f_{12} - f_0 f_2 (D_t f_1 \cdot f_{12}) = {} & -3\lambda_1^2 [(D_x f_0 \cdot f_2) f_1 f_{12} - f_0 f_2 (D_x f_1 \cdot f_{12})] \\ & + 3\lambda_1 \left[\left(D_x^2 f_0 \cdot f_1\right) f_2 f_{12} - f_0 f_1 \left(D_x^2 f_2 \cdot f_{12}\right]\right. \\ & - \left(D_x^3 f_0 \cdot f_1\right) f_2 f_{12} + f_0 f_1 \left(D_x^3 f_2 \cdot f_{12}\right).\end{aligned} \tag{16}$$

Similarly, using equations (12) and (13), we have

$$\begin{aligned}(D_t f_1 \cdot f_{12}) f_0 f_2 - f_1 f_{12} (D_t f_0 \cdot f_2) = {} & -3\lambda_2^2 [(D_x f_1 \cdot f_{12}) f_0 f_2 - f_1 f_{12} (D_x f_0 \cdot f_2)] \\ & + 3\lambda_2 \left[\left(D_x^2 f_1 \cdot f_{12}\right) f_0 f_2 - f_1 f_{12} \left(D_x^2 f_0 \cdot f_2\right]\right. \\ & - \left(D_x^3 f_1 \cdot f_{12}\right) f_0 f_2 + f_1 f_{12} \left(D_x^3 f_0 \cdot f_2\right).\end{aligned} \tag{17}$$

Then adding equation (16) with equation (17), we have

$$\begin{aligned}3\left(\lambda_1^2 - \lambda_2^2\right) [(D_x f_0 \cdot f_2) f_1 f_{12} - f_0 f_2 (D_x f_1 \cdot f_{12})] = {} & 3\lambda_1 \left[\left(D_x^2 f_0 \cdot f_1\right) f_2 f_{12}\right. \\ & \left. - f_0 f_1 \left(D_x^2 f_2 \cdot f_{12}\right)\right] + 3\lambda_2 \left[\left(D_x^2 f_1 \cdot f_{12}\right) f_0 f_2 - f_1 f_{12} \left(D_x^2 f_0 \cdot f_2\right)\right] \\ & - (D_x^3 f_0 \cdot f_1) f_2 f_{12} + f_0 f_1 \left(D_x^3 f_2 \cdot f_{12}\right) \\ & - \left(D_x^3 f_1 \cdot f_{12}\right) f_0 f_2 + f_1 f_{12} \left(D_x^3 f_0 \cdot f_2\right).\end{aligned} \tag{18}$$

Taking account of (A.7) and (A.8), we find

$$\begin{aligned}\left(\lambda_1^2 - \lambda_2^2\right) [(D_x f_0 \cdot f_2) f_1 f_{12} - f_0 f_2 (D_x f_1 \cdot f_{12})] = {} & D_x [(\lambda_1 - \lambda_2)(D_x f_0 \cdot f_{12}) \cdot f_2 f_1] \\ & + D_x [(\lambda_1 + \lambda_2) f_0 f_{12} \cdot (D_x f_2 \cdot f_1)] + D_x [(D_x f_0 \cdot f_{12}) \cdot (D_x f_1 \cdot f_2)],\end{aligned} \tag{19}$$

or

$$D_x \{[(D_x - \lambda_1 - \lambda_2) f_0 \cdot f_{12}] \cdot [(D_x + \lambda_1 - \lambda_2) f_1 \cdot f_2]\} = 0, \tag{20}$$

which implies that

$$(D_x - \lambda_1 - \lambda_2) f_0 \cdot f_{12} = c_1 (D_x + \lambda_1 - \lambda_2) f_1 \cdot f_2, \tag{21}$$

where $c_1 = c_1(t)$ is an integration constant.

Now we consider the other part. With the assumption that (21) holds, we consider

$$\mathbb{Q} \equiv [(SD_x - \lambda_1 S) f_0 \cdot f_1] f_2 - [(SD_x - \lambda_2 S) f_0 \cdot f_2] f_1,$$

it yields

$$\begin{aligned}
\mathbb{Q} &\overset{(A.9-A.10)}{=} -(\mathcal{D}f_0)(D_x f_1\cdot f_2) - f_{0x}(Sf_1\cdot f_2) + \frac{1}{2}f_0\left[(Sf_0\cdot f_1)_x f_2 + \mathcal{D}(D_x f_1\cdot f_2)\right]\\
&\quad - (\lambda_1-\lambda_2)\left[(\mathcal{D}f_0)f_1f_2 - \frac{1}{2}f_0(\mathcal{D}f_1f_2)\right] + \frac{1}{2}(\lambda_1+\lambda_2)f_0(Sf_1\cdot f_2)\\
&= -(\mathcal{D}f_0)\left[(D_x+\lambda_1-\lambda_2)f_1\cdot f_2\right] + \frac{1}{2}f_0\mathcal{D}\left[(D_x+\lambda_1-\lambda_2)f_1\cdot f_2\right]\\
&\quad - f_{0x}(Sf_1\cdot f_2) + \frac{1}{2}f_0(Sf_1\cdot f_2)_x + \frac{1}{2}(\lambda_1+\lambda_2)f_0(Sf_1\cdot f_2)\\
&\overset{(21)}{=} \underbrace{-\frac{(\mathcal{D}f_0)}{c_1}\left[(D_x-\lambda_1-\lambda_2)f_0\cdot f_{12}\right] + \frac{f_0}{2c_1}\mathcal{D}\left[(D_x-\lambda_1-\lambda_2)f_0\cdot f_{12}\right]}\\
&\quad - f_{0x}(Sf_1\cdot f_2) + \frac{1}{2}f_0(Sf_1\cdot f_2)_x + \frac{1}{2}(\lambda_1+\lambda_2)f_0(Sf_1\cdot f_2).
\end{aligned} \tag{22}$$

We now note that the underbraced terms can be reformulated as

$$-\frac{f_{0x}}{c_1}(Sf_0\cdot f_{12}) + \frac{f_0}{2c_1}\left[(Sf_0\cdot f_{12})_x + (\lambda_1+\lambda_2)Sf_0\cdot f_{12}\right],$$

therefore we have

$$\mathbb{Q} = \left[-f_{0x} + \frac{1}{2}f_0\frac{\partial}{\partial x} + \frac{1}{2}(\lambda_1+\lambda_2)f_0\right]\left[\frac{1}{c_1}Sf_0\cdot f_{12} + Sf_1\cdot f_2\right]. \tag{23}$$

From $\mathbb{Q} = 0$ we have

$$\frac{1}{c_1}Sf_0\cdot f_{12} + Sf_1\cdot f_2 = c_2 f_0^2\,\mathrm{e}^{-(\lambda_1+\lambda_2)x} \tag{24}$$

where $c_2 = c_2(t)$ is an integration constant.

The nonlinear superposition formula constitutes equations (21) and (24), which are differential equations to be solved to find solutions. Interestingly, they can be solved in a closed form

$$\begin{aligned}
f_{12} = \frac{c_1}{(\lambda_1+\lambda_2)f_0}\Bigg[&\frac{2(\mathcal{D}f_0)Sf_1\cdot f_2}{f_0} - 2D_x f_1\cdot f_2 + 2(\mathcal{D}f_1)(\mathcal{D}f_2)\\
&- (\lambda_1-\lambda_2)f_1f_2 + c_2 f_0^2(\mathcal{D}\,\mathrm{e}^{-(\lambda_1+\lambda_2)x})\Bigg].
\end{aligned}$$

However, by setting $c_1 = 1$ and $c_2 = 0$, we have

$$f_{12} = \frac{1}{(\lambda_1+\lambda_2)f_0}\left[\frac{2(\mathcal{D}f_0)Sf_1\cdot f_2}{f_0} - 2D_x f_1\cdot f_2 + 2(\mathcal{D}f_1)(\mathcal{D}f_2) - (\lambda_1-\lambda_2)f_1f_2\right],$$

this nonlinear superposition formula is of differential-algebraic nature.

Let us now try to calculate some solutions. As usual, we start with the trivial solution

$$f_0 = 1,$$

then we find the 1-soliton solution

$$f = 1 + \mathrm{e}^{\eta}, \qquad \eta = kx - k^3t + \theta\xi,$$

where k is the ordinary wavenumber which relates to the spectral parameter through $k = -\lambda$ and ξ is a Grassmann odd constant. To construct a 2-soliton, we take

$$f_1 = 1 + \mathrm{e}^{\eta_1}, \qquad f_2 = 1 + \mathrm{e}^{\eta_2},$$

then we obtain

$$f_{12} = -\frac{k_1 - k_2}{k_1 + k_2} + e^{\eta_1} - e^{\eta_2} + \frac{k_1 - k_2 - 2\xi_1\xi_2 - 2\theta(k_1\xi_2 - k_2\xi_1)}{k_1 + k_2} e^{\eta_1+\eta_2}.$$

We may continue to calculate a 3-soliton solution, which is

$$f_{123} = s_0 + s_1 e^{\eta_1} + s_2 e^{\eta_2} + s_3 e^{\eta_3} + s_{12} e^{\eta_1+\eta_2} + s_{13} e^{\eta_1+\eta_3} + s_{23} e^{\eta_2+\eta_3} + s_{123} e^{\eta_1+\eta_2+\eta_3}, \tag{25}$$

where

$$s_0 = \frac{(k_1 - k_2)(k_1 - k_3)(k_2 - k_3)}{(k_1 + k_2)(k_1 + k_3)(k_2 + k_3)}, \qquad s_1 = \frac{k_2 - k_3}{k_2 + k_3}, \qquad s_2 = -\frac{k_1 - k_3}{k_1 + k_3},$$

$$s_3 = \frac{k_1 - k_2}{k_1 + k_2}, \qquad s_{12} = \frac{-k_1 + k_2 + 2m_{12}}{k_1 + k_2}, \qquad s_{13} = \frac{k_1 - k_3 - 2m_{13}}{k_1 + k_3},$$

$$s_{23} = \frac{-k_2 + k_3 + 2m_{23}}{k_2 + k_3},$$

$$s_{123} = -s_0 + 2\frac{(m_{12} + m_{23} - m_{13})(k_1k_2 + k_1k_3 + k_2k_3) + \left(m_{12}k_3^2 + m_{23}k_1^2 - m_{13}k_2^2\right)}{(k_1 + k_2)(k_1 + k_3)(k_2 + k_3)},$$

and

$$m_{ij} = \xi_i\xi_j + \theta(k_i\xi_j - k_j\xi_i).$$

These solutions are equivalent to those found early in [2].

5. Conclusions

In this paper, we have shown that on the basis of Hirota's direct method, we may extract a lot of properties for the sKdV equation. We derive a BT, a Lax representation, and a modified equation. We also constructed a nonlinear superposition formula within the bilinear approach. Thus, even in the supersymmetric case, Hirota's bilinear method is a very effective method.

In the classical case, the chain of a BT supplies a chain of integrable systems. In the KdV case, such a situation is studied by Nakamura and Hirota [14, 15]. It would be interesting to find out what happens in the supersymmetric case.

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Appendix. Some bilinear identities

In this appendix, we list the relevant bilinear identities, which can be proved directly. Here a, b, c and d are arbitrary even functions of the independent variables x, t and θ.

$$(SD_ta \cdot a)b^2 - a^2(SD_tb \cdot b) = 2S(D_ta \cdot b) \cdot ba \tag{A.1}$$

$$\begin{aligned}\left(SD_x^3a \cdot a\right)b^2 - a^2\left(SD_x^3b \cdot b\right) = 2S\left[\left(D_x^3a \cdot b\right) \cdot ab + 3\left(D_x^2a \cdot b\right) \cdot (D_xb \cdot a)\right] \\ - 3\left[(SD_xa \cdot a)\left(D_x^2b \cdot b\right) - (SD_xb \cdot b)\left(D_x^2a \cdot a\right)\right]\end{aligned} \tag{A.2}$$

$$S\left(D_x^2 a\cdot b\right)\cdot(D_x b\cdot a)-D(SD_x a\cdot b)\cdot(D_x b\cdot a)$$
$$=\tfrac{1}{2}\left[(SD_x a\cdot a)\left(D_x^2 b\cdot b\right)-(SD_x b\cdot b)\left(D_x^2 a\cdot a\right)\right] \qquad \text{(A.3)}$$

$$D_x(Sa\cdot b)\cdot(D_x b\cdot a)=D_x(SD_x a\cdot b)\cdot ba-S(D_x^2 a\cdot b)\cdot ab \qquad \text{(A.4)}$$

$$S(D_x a\cdot b)\cdot ba=D_x(Sa\cdot b)\cdot ba \qquad \text{(A.5)}$$

$$(D_x a\cdot b)cd-ab(D_x c\cdot d)=(D_x a\cdot c)bd-ac(D_x b\cdot d) \qquad \text{(A.6)}$$

$$\left(D_x^2 a\cdot b\right)cd-ab(D_x^2 c\cdot d)=D_x\left[(D_x a\cdot d)\cdot cb+ad\cdot(D_x c\cdot b)\right] \qquad \text{(A.7)}$$

$$\left(D_x^3 a\cdot b\right)cd+ab\left(D_x^3 c\cdot d\right)=\left(D_x^3 a\cdot d\right)cb+ad\left(D_x^3 c\cdot b\right)-3D_x(D_x a\cdot c)\cdot(D_x b\cdot d), \qquad \text{(A.8)}$$

$$(SD_x a\cdot b)c-(SD_x a\cdot c)b=-(\mathcal{D}a)(D_x b\cdot c)-a_x(Sb\cdot c)+\tfrac{1}{2}a\left[(Sb\cdot c)_x+\mathcal{D}(D_x b\cdot c)\right], \qquad \text{(A.9)}$$

$$\lambda_1(Sa\cdot b)c-\lambda_2(Sa\cdot c)b=(\lambda_1-\lambda_2)(\mathcal{D}a)bc-\tfrac{1}{2}a\left[(\lambda_1+\lambda_2)(Sb\cdot c)+(\lambda_1-\lambda_2)\mathcal{D}bc\right]. \qquad \text{(A.10)}$$

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